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LETTER TO THE EDITOR

Exact solutions of a class of nonlinear boundary value problems with moving boundaries

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Processes of melting and evaporation of metals in the case that their surface is exposed to a powerful flux of energy, are known to be described by a nonlinear boundary value problem [1, 2]:

$$\frac{\partial}{\partial x}\left(\lambda_K(T_K)\frac{\partial T_K}{\partial x}\right)=C_K(T_K)\frac{\partial T_K}{\partial \tau} \qquad K=\mathrm{s},\mathrm{l} \tag{1}$$

$$x=S_\mathrm{l}(\tau):\lambda_\mathrm{l}(T_\mathrm{l})\frac{\partial T_\mathrm{l}}{\partial x}=v_\mathrm{l}H_v-q \qquad T_\mathrm{l}=T_v \tag{2}$$

$$x=S_\mathrm{s}(\tau):\lambda_\mathrm{s}(T_\mathrm{s})\frac{\partial T_\mathrm{s}}{\partial x}=\lambda_\mathrm{l}(T_\mathrm{l})\frac{\partial T_\mathrm{l}}{\partial x}+v_\mathrm{s}H_m \qquad T_\mathrm{s}=T_\mathrm{l}=T_m \tag{3}$$

$$T_\mathrm{s}|_{x\to\infty}=T_0 \qquad T_K|_{\tau=0}=T_0 \qquad K=\mathrm{s},\mathrm{l} \tag{4}$$

where T_m, T_v, T_0 are the temperatures of melting and evaporation, and the original temperature, respectively; λ_K are thermal conductivities; H_m, H_v, C_K are specific heat values per unit volume; q is the energy flux being absorbed by the metal; S_K are the phase division boundary coordinates to be found; v_K are the phase division boundary velocities to be found; T_K are the temperature fields to be found; indices s and l corresponding to solid and liquid phases, respectively.

In a quasistationary approxation in the linear case the problem of (1)–(4) is solved in [3]. In the case when $C_K(T_K)$ or $\lambda_K(T_K)$ are linear functions, this problem has been solved in [2]. In this report a method for construction of the exact solutions of the problem in the case of arbitrary, sufficiently smooth functions of λ_K, C_K, $K=\mathrm{s},\mathrm{l}$, is suggested, the illustrations for some specific nonlinearities being given.

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It is known that after a short transient of melting and evaporation there occurs a quasistationary phase for which the relations for velocities are $v_s = v_l = v = \text{constant}$ and for the melt thickness $S_s - S_1 = \delta = \text{constant}$. This enables us to use in the problem (1)–(4) a moving coordinate system attached to the evaporation front according to the law $\xi = x - v\tau$, hence the functions to be found have the form $T_K(\tau, x) = T_K(\xi)$. Thus, the initial problem is reduced to the boundary value problem for the following ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\lambda_K(T_K)\frac{\mathrm{d}T_K}{\mathrm{d}\xi}\right) + vC_K(T_K)\frac{\mathrm{d}T_K}{\mathrm{d}\xi} = 0 \qquad K = \mathrm{s}, \mathrm{l} \tag{5}$$

$$\xi = 0\colon \lambda_l(T_l)\frac{\mathrm{d}T_l}{\mathrm{d}\xi} = vH_v - q \qquad T_l = T_v \tag{6}$$

$$\xi = \delta\colon \lambda_s(T_s)\frac{\mathrm{d}T_s}{\mathrm{d}\xi} = \lambda_l(T_l)\frac{\mathrm{d}T_l}{\mathrm{d}\xi} + vHm \qquad T_s = T_l = T_m \tag{7}$$

$$T_s|_{\xi\to\infty} = T_0. \tag{8}$$

The nonlinear equations (5) at $C_K(T_K) = 0$ are well known to be linearized, even in the multidimensional case, by the substitution $W_K = \int \lambda_K(T_K)\mathrm{d}T_K$. This substitution can apparently be generalized as well to the case $C_K(T_K) \neq 0$ by introducing a new independent variable η

$$\begin{aligned} z_s &= \int_{T_0}^{T_s} C_s(T)\mathrm{d}T \equiv W_s(T_s) \\ z_l &= \int_{T_m}^{T_l} C_l(T)\mathrm{d}T \equiv W_l(T_l) \end{aligned} \tag{9}$$

$$\begin{aligned} \xi &= \int_0^{\eta} \frac{\lambda_l}{C_l}(w_l^{-1}(z_l(\eta)))\mathrm{d}\eta \qquad 0 \leqslant \xi \leqslant \delta \\ \xi &= \int_{\delta^*}^{\eta} \frac{\lambda_s}{C_s}(W_s^{-1}(z_s(\eta)))\mathrm{d}\eta + \delta \qquad \xi \geqslant \delta \end{aligned} \tag{10}$$

where W_K^{-1} are inverse functions to $W_K(T_K)$, $K = \mathrm{s}, \mathrm{l}$, and the lower bounds of integration for convenience are chosen to fit $W_s(T_0) = 0$, $W_l(T_m) = 0$, $\xi|_{\eta=0} = 0$, $\xi|_{\eta=\delta^*} = \delta$.

Since the values $C_K(T_K)$, $\lambda_K(T_K)$ according to their physical meaning are continuous and positive, there are strictly monotonic increasing functions z_K and ξ (depending on T_K and η, respectively) on the RHS of (9) and (10).

Hence there exist inverse functions to W in the intervals shown. Homeomorphisms between the functions and variables $z_s(\eta)$, $z_l(\eta)$, η and $T_s(\xi)$, $T_l(\xi)$, ξ, respectively, can thus be established. Having substituted variables of (9) and (10) into the equations and boundary conditions (5)–(8), after simple transformation, a linear problem is obtained:

$$\frac{d^2z_K}{d\eta^2}+v\frac{dz_K}{d\eta}=0 \qquad K=s,l \tag{11}$$

$$\eta=0: \frac{dz_l}{d\eta}=vH_v-q \qquad z_l=W_l(T_v)\equiv T_v^* \tag{12}$$

$$\eta=\delta^*: \frac{dz_s}{d\eta}=\frac{dz_l}{d\eta}+vH_m \qquad z_s=W_s(T_m)\equiv T_m^* \qquad z_l=0 \tag{13}$$

$$z_s|_{\eta\to\infty}=0 \tag{14}$$

where $z_K(v)$, v, δ^* are unknown functions and constants.

After the problem of equations (11)–(14) has been solved, we obtain

$$z_s(\eta)=T_m^*\exp v(\delta^*-\eta) \qquad \eta\geqslant\delta^* \tag{15}$$

$$z_l(\eta)=T_v^*\frac{\exp(v(\delta^*-\eta))-1}{e^{v\delta^*}-1} \qquad 0\leqslant\eta\leqslant\delta^* \tag{16}$$

$$v=q(T_v^*+T_m^*+H_m+H_v)^{-1} \tag{17}$$

$$\delta^*=\frac{1}{v}\ln\left(1+\frac{T_v^*}{T_m^*+H_m}\right). \tag{18}$$

Thus, to obtain the solution of the nonlinear boundary value problem (5)–(8) one should only substitute the expressions (15)–(18) into (9) and (10). Then an implicit solution of the problem (5)–(8), and, consequently, the solution of the original nonlinear problem (1)–(4) in a quasistationary approximation, is obtained. To obtain the expressions for the functions $T_K(\xi)=T_K(x-v\tau)$ in an explicit form one should, possessing specific functions λ_K, C_K, $K=s,l$, solve for the functional relations with respect to T_K and δ. Particularly, in the case of constant values λ_K, C_K the results of [3] can be easily obtained.

Note 1. For the velocity v, (17) gives the value in an explicit form, since T_m^*, T_v^* are determined from (12) and (13).

Below we state the results yielded by the substitutions of (9) and (10) in some specific cases of temperature dependence exhibited by λ_K and C_K, $K=\mathrm{s, l}$.

Case 1. If $\lambda_K(T_K)=\lambda_K$, $C_K(T_K)=c_K+d_K T_K$, $\lambda_K, c_K, d_K \in \mathbb{R}^1$, then

$$T_s=\frac{T_0 C_s\left(\frac{T_m+T_0}{2}\right)+(T_m-T_0)C_s\left(\frac{T_0}{2}\right)\exp\left[-v\frac{C_s(T_0)}{\lambda_s}(\xi-\delta)\right]}{C_s\left(\frac{T_m+T_0}{2}\right)-\frac{1}{2}d_s(T_m-T_0)\exp\left[-v\frac{C_s(T_0)}{\lambda_s}(\xi-\delta)\right]}\qquad \xi\geqslant\delta \tag{19}$$

$$T_l=\frac{K}{d_l}\frac{C_l(T_v)+K+(C_l(T_v)-K)\exp\left(-v\frac{K}{\lambda_l}\xi\right)}{C_l(T_v)+K-(C_l(T_v)-K)\exp\left(-v\frac{K}{\lambda_l}\xi\right)}-\frac{C_l}{d_l}\qquad 0\leqslant\xi\leqslant\delta \tag{20}$$

$$\delta=\frac{\lambda_l}{vK}\ln\left|\frac{(C_l(T_v)-K)(C_l(T_m)+K)}{(C_l(T_v)+K)(C_l(T_m)-K)}\right| \tag{21}$$

where

$$K=\left[C_l^2(T_m)-2d_l H_m-2d_l(T_m-T_0)C_s\left(\frac{T_m+T_0}{2}\right)\right]^{1/2}.$$

Note 2. In the case $c_s=0$ equations (18)–(21) coincide with the corresponding formulae from [2] (in the latter the conditon $c_s=0$ having been accidentaly omitted).

Case 2. If $\lambda_K(T_K)=\lambda_K$, $C_K(T_K)=\mathrm{e}^{\alpha_K T_K}$, $\lambda_K, \alpha_K \in \mathbb{R}^1$, $K=\mathrm{s, l}$, then

$$T_s=T_0-\frac{1}{\alpha_s}\ln\left[1-(1-\mathrm{e}^{-\alpha_s(T_m-T_0)})\exp\left[-v\frac{\mathrm{e}^{\alpha_s T_0}}{\lambda_s}(\xi-\delta)\right]\right] \tag{22}$$

$$T_l=T_v-\frac{1}{\alpha_l}\ln\left[\frac{\mathrm{e}^{\alpha_l T_v}(\mathrm{e}^{v\delta^*}-1)\exp\left(\frac{\mathrm{e}^{v\delta^*+\alpha_l T_m}-\mathrm{e}^{\alpha_l T_v}}{\mathrm{e}^{v\delta^*}-1}\cdot\frac{v}{\lambda_l}\xi\right)-\mathrm{e}^{v\delta^*}(\mathrm{e}^{\alpha_l T_v}-\mathrm{e}^{\alpha_l T_m})}{(\mathrm{e}^{v\delta^*+\alpha_l T_m}-\mathrm{e}^{\alpha_l T_m})\exp\left(\frac{\mathrm{e}^{v\delta^*+\alpha_l T_m}-\mathrm{e}^{\alpha_l T_v}}{\mathrm{e}^{v\delta^*}-1}\cdot\frac{v}{\lambda_l}\xi\right)}\right] \tag{23}$$

$$\delta=\frac{\lambda_l}{v\mathrm{e}^{\alpha_l T_v}}\cdot\frac{\mathrm{e}^{v\delta^*}-1}{1-\exp(-v\delta^*+\alpha_l(T_v-T_m))} \tag{24}$$

where

$$e^{v\delta^*}=1+\frac{\alpha_l^{-1}(e^{\alpha_l T_v}-e^{\alpha_l T_m})}{\alpha_s^{-1}(e^{\alpha_s T_m}-e^{\alpha_s T_0})+H_m}.$$

Case 3. If $\lambda_K=a_K+b_K T_K^2$, $C_K(T_K)=c_K$, $K=$s, l, a_K, b_K, $c_K\in\mathbb{R}^1$ then

$$T_s=T_0+(T_m-T_0)e^{v(\delta^*-\eta(\xi))} \qquad \xi\geqslant\delta \tag{25}$$

$$T_l=T_m+\frac{T_v-T_m}{e^{v\delta^*}-1}(e^{v(\delta^*-\eta(\xi))}-1) \qquad 0\leqslant\xi\leqslant\delta \tag{26}$$

where

$$e^{v\delta^*}=1+\frac{c_l(T_v-T_m)}{c_s(T_m-T_0)+H_m}$$

the function $\eta(\xi)$ being a solution of the transcendental equations

$$(\eta-\delta^*)\frac{a_s+b_sT_0^2}{c_s}-\frac{b_s(T_m-T_0)^2}{2vc_s}e^{2v\delta^*}(e^{-v\eta}-e^{-v\delta^*})\left[\left(\frac{4T_0}{T_m-T_0}+1\right)e^{-v\delta^*}+e^{-v\eta}\right]$$
$$=\xi-\delta \qquad \xi\geqslant\delta \tag{27}$$

$$\left[\frac{a_l+b_lT_m^2}{b_l}-\frac{c_l^2}{(c_s(T_m-T_0)+H_m)^2}-\frac{2T_mc_l}{c_s(T_m-T_0)+H_m}+1\right]\eta$$
$$+\frac{2}{v}\left(1-\frac{T_mC_l}{c_s(T_m-T_0)+H_m}\right)e^{v(\delta^*-\eta)}-\frac{1}{2v}e^{2v(\delta^*-\eta)}$$
$$=\frac{c_l^3\xi}{b_l(c_s(T_m-T_0)+H_m)^2} \qquad O\leqslant\xi\leqslant\delta. \tag{28}$$

By expanding the LHS of (27) and (28) into a Taylor series in the vicinity of $\eta=\delta^*$ and by considering only powers of η less than 3, the explicit form of $\eta(\xi)$ in the vicinity of $\xi=\delta$ is easily obtained. The expressions obtained are omitted as being too cumbersome.

Case 4. If $\lambda_k(T_k)=\lambda_kT_k^{\beta_k}$, $c_k(T_k)=c_kT_k^{\alpha_k}$, λ_k, c_k, α_k, $\beta_k\in\mathbb{R}^1$, then

$$T_s=T_m\left[1+\frac{c_s\gamma_s v}{\lambda_s}T_m^{\alpha_s-\beta_s}(\xi-\delta)\right]^{1/(\alpha+1)} \qquad \xi\geqslant\delta$$

$$T_l=\left(T_v^{1+\alpha_l}+\frac{c_l\gamma_l v}{\lambda_l}\xi T^{2\alpha_l+1-\beta_l}\right)^{1/(\alpha_l+1)} \qquad 0\leqslant\xi\leqslant\delta$$

$$\delta=\frac{\lambda_l}{vc_l\gamma_l}T_m^{\beta_l-2\alpha_l-1}(T_m^{\alpha_l+1}-T_v^{\alpha_l+1})$$

where it is assumed that

$$T_0=O \qquad H_n=\frac{c_l}{\alpha_l+1}T_m^{\alpha_l+1}-\frac{c_s}{\alpha_s+1}T_m^{\alpha_l+1} \qquad \gamma_k=\frac{\alpha_k-\beta_k}{\alpha_k+1} \qquad \alpha_k<\beta_k$$

$$\alpha_s<-1 \qquad \alpha_l\neq-1 \qquad k=\text{s, l}.$$

Concluding, we should note that the procedure of returning from the solution of the initial problem in an implicit form of (15)–(18) to the solution in the explicit form can be realized also for $C_k(T_k) = c_k T_k^{-1}$, $\lambda_k(T_k) = \lambda_k T_k^{\beta_k}$ nonlinearities, where λ_k, c_k, $\beta_k \in \mathbb{R}^1$. The suggested linearization method of the boundary value problem equations (1)–(4) holds also if a more complicated condition [4] holds instead of the condition at the evaporation front, $T_l = T_v$.

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